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A note on wild fibers of elliptic surfaces

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0. Introduction

Let k be an algebraically closed field of characteristic p, and $f: S \to C$ an elliptic surface over k with C a non-singular complete curve. Assume that $f^{-1}(P) = dE$ $(P \in C)$ is a multiple fiber with multiplicity d. The multiple fiber is called a tame fiber (resp. a wild fiber) if the order of the normal bundle $\mathcal{O}_S(E)|_E$ is equal to d (resp. less than d). In characteristic 0, there does not exist a wild fiber by the cohomological flatness. In positive characteristic, however, the existence of wild fibers makes the situation complicated. The notion of wild fiber was introduced in Bombieri and Mumford [1], and Raynaud [5] examined the structure of wild fiber in detail. In this note, we consider elliptic surfaces obtained as quotients of the product of a curve and a supersingular elliptic curve by rational vector fields in positive characteristic. We calculate numerical invariants of wild fibers of such elliptic surfaces (cf. Theorem 3.5). Moreover, we give a characterization of such elliptic surfaces over the projective line $\mathbb{P}^1$ (cf. Theorem 4.2). To calculate numerical invariants, Raynaud's results on wild fibers play an important role (cf. [5]). For the case of the product of a curve and an ordinary elliptic curve, we already treated this in [3].

1. Preliminaries

In this section, we recall some basic facts on elliptic surfaces and Raynaud's theory on wild fibers. For details, see Bombieri and Mumford [1] and Raynaud [5].

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Throughout this paper, we fix an algebraically closed field k of characteristic $p > 0$. For a non-singular complete algebraic variety X of dimension n and a coherent sheaf $\mathscr{F}$ on X, we use the following notation:

$\mathcal{O}_X$: the structure sheaf of X,
K_X: a canonical divisor on X,
$b_i(X)$: the ith Betti number of X,
$c_n(X)$: the nth Chern number of X,
$q(X)$: the dimension of Albanese variety $\mathrm{Alb}(X)$ of X,
$\mathrm{Pic}^0(X)$: the Picard scheme of X,
$H^i(X, \mathscr{F})$: the ith cohomology group with coefficients in $\mathscr{F}$,
$\chi(X, \mathcal{O}_X) = \sum_{i=0}^{n}(-1)^i \dim_k H^i(X, \mathcal{O}_X)$,
$\mathrm{Supp}\,\mathscr{F}$: the support of $\mathscr{F}$,
$\Gamma(U, \mathscr{F})$: the group of sections of $\mathscr{F}$ over an open set U of X.

For divisors E_1 and E_2 on X, $E_1 \sim E_2$ means that E_1 is linearly equivalent to E_2. Sometimes, a Cartier divisor and the associated invertible sheaf will be identified. For a rational number x, $[x]$ denotes the integral part of x.

Now, let $f: S \to C$ be an elliptic surface defined over k with C a non-singular complete curve. We assume that $f: S \to C$ is relatively minimal, i.e., no fibers of f contain exceptional curves of the first kind. Let $\mathscr{T}$ be the torsion part of $R^1 f_* \mathcal{O}_S$. Since C is a non-singular curve, we have $R^1 f_* \mathcal{O}_S \simeq \mathscr{L} \oplus \mathscr{T}$ with an invertible sheaf $\mathscr{L}$. We denote by $d_i E_i$ $(i = 1, 2, \dots, \lambda)$ the multiple singular fibers of $f: S \to C$ with multiplicities d_i. We have the canonical divisor formula,

$$K_S \sim f^*(K_C - \mathscr{L}) + \sum_{i=1}^{\lambda} a_i E_i, \tag{1.1}$$

where a_i's are integers such that $0 \le a_i \le d_i - 1$, and where

$$-\deg \mathscr{L} = \chi(S, \mathcal{O}_S) + t \quad \text{with } t = \text{length } \mathscr{T}. \tag{1.2}$$

We take a multiple fiber dF among $d_i E_i$'s $(i = 1, 2, \dots, \lambda)$, and set $Q = f(dF)$. We denote by a the a_i corresponding to the multiple fiber dF. We denote by $\omega_{S/C}$ the relative dualizing sheaf on S. We can naturally consider nF as a subscheme of S. The dualizing sheaf ω_n of nF is given by

$$\omega_n = \omega_{S/C} \otimes \mathcal{O}_S(nF)|_{nF}. \tag{1.3}$$

We denote by v the order of $\mathcal{O}_S(E)|_E$. Then, there exists a positive integer γ such that

$$d = vp^{\gamma} \tag{1.4}$$

(cf. [5, Lemma 3.7.7]). Using the exact sequence

$$0 \to \mathcal{O}_S(-nF) \to \mathcal{O}_S \to \mathcal{O}_{nF} \to 0,$$

we see $\chi(\mathcal{O}_{nF}) = 0$. Therefore, by Raynaud [5, Corollary 3.7.6] and the Serre duality, we have the following:

Lemma 1.1 (Raynaud). *Assume $n \geq 2$. If ω_n is not trivial, then*

$$\dim_k H^0(nF, \mathcal{O}_{nF}) = \dim_k H^0((n-1)F, \mathcal{O}_{(n-1)F}).$$

If ω_n is trivial, then

$$\dim_k H^0(nF, \mathcal{O}_{nF}) = \dim_k H^0((n-1)F, \mathcal{O}_{(n-1)F}) + 1.$$

Lemma 1.2 (Raynaud [5, Lemma 3.7.7]). *Assume $n \geq 2$. Then, we have either $\mathrm{ord}(\mathcal{O}_S(F)|_{nF}) = ord(\mathcal{O}_S(F)|_{(n-1)F})$ or $\mathrm{ord}(\mathcal{O}_S(F)|_{nF}) = p\,\mathrm{ord}(\mathcal{O}_S(F)|_{(n-1)F})$. In the latter case, ω_n is trivial.*

We denote by n_i ($i = 0, 1, \ldots, \gamma$) the smallest integer n such that $\mathcal{O}_S(F)|_{nF}$ is of order vp^i. By definition, we have $n_0 = 1$, and by Lemma 1.2, ω_{n_i} ($i = 0, 1, \ldots, \gamma$) is trivial. Using the notation above, we have the following:

Lemma 1.3 (Raynaud [5, Lemma 3.7.9, p. 31]). (i) *There exists an integer $k_i > 0$ such that $n_{i+1} = n_i + k_i v p^i$.*

(ii) *There exists a positive integer h such that $n_\gamma = mh - a$.*

Lemma 1.4 (Raynaud [5, Theorem 3.8.1, p. 32]). *The length of $\mathcal{T}$ at a point Q of C is given by*

$$\text{length } \mathcal{T}_Q = [\chi/d],$$

where $\chi = d\{(1 - 1/d) + k_0(1 - 1/p^\gamma) + \cdots + k_{\gamma-1}(1 - 1/p)\}$.

Corollary 1.5. *Assume $d = p$ and $v = 1$. Then,*

$$\text{length } \mathcal{T}_Q = [n_1(p-1)/p].$$

Proof. By assumption, we have $\gamma = 1$. By Lemma 1.3(i), we have $n_1 = 1 + k_0$. Therefore, by Lemma 1.4 we have

$$\chi = p\{(1 - 1/p) + (n_1 - 1)(1 - 1/p)\} = n_1(p-1). \quad \square$$

2. Rational vector fields

Let D be a non-zero rational vector field on a non-singular algebraic variety X. D is called a p-closed vector field if there exists a rational function f on X such that $D^p = fD$. In particular, D is called additive if $D^p = 0$. For an affine open covering $\{\mathrm{Spec}\, A_i\}_{i \in I}$ of X, we set $A_i^D = \{\alpha \in A_i \mid D(\alpha) = 0\}$ for each $i \in I$. Then, $\mathrm{Spec}\, A_i^D$ ($i \in I$) glue together to define a quotient surface X^D. It is well known that X^D is normal. If

D is p-closed, then the canonical projection $\pi_D : X \to X^D$ is a finite purely inseparable morphism of degree p. Conversely, if $\pi : X \to Y$ is a finite purely inseparable morphism of degree p with a normal variety Y, then there exists a p-closed rational vector field D such that $\pi = \pi_D$ and $Y = X^D$. Moreover, X^D is non-singular if and only if D has no isolated singularities. We denote by (D) the divisor associated with D. For details on these facts, see, for example, Rudakov and Shafarevich [6].

Lemma 2.1. *Let E be a supersingular elliptic curve and δ a non-zero regular vector field on E. Then, for any point Q of E, there exists a rational function f on E such that $\delta(f) = 1$ and such that f is regular at Q.*

Proof. Since E is supersingular, the vector field δ is additive. By Ganong and Russell [2, Lemma 3.3.1], there exists a rational function h such that $\delta(h) = 1$. If h is regular at Q, then we can take $f = h$. Assume that h has a pole at Q. Then, we take a point P of E such that h is regular at P. There exists a translation T of E such that $T(Q) = P$. We set $f = T^*(h)$. Then, f is regular at Q. Since δ is invariant under translation, we have

$$\delta(f) = \delta(T^*h) = (T_*\delta)h = \delta(h) = 1. \quad \square$$

Now, let C be a non-singular complete curve of genus g. Let P be a point of C, and x a local parameter at P. We take a rational vector field $\Delta = h(\partial/\partial x)$ of C, where h is a non-zero rational function on C. We set

$$\Delta^i = \sum_{j=1}^{i} h_{ij} \frac{\partial^j}{\partial x^j}. \tag{2.1}$$

Then, by definition, we have

$$h_{11} = h. \tag{2.2}$$

Moreover, denoting $\partial/\partial x$ by $'$, we have for $i \geq 2$

$$h_{i1} = h_{11} h'_{i-1,1},$$

$$h_{ij} = h_{11}\{h_{i-1,j-1} + h'_{i-1,j}\} \quad (1 < j < i),$$

$$h_{ii} = h_{11} h_{i-1,i-1} = h^i. \tag{2.3}$$

In characteristic $p > 0$, we have

$$h_{pj} = 0 \quad (2 \leq j \leq p-1) \quad \text{and} \quad h_{pp} \frac{\partial^p}{\partial x^p} = 0. \tag{2.4}$$

Therefore, as is well known, Δ is additive if and only if $h_{p1} = 0$.

Let P be a zero point of h, and x a local parameter at P. Then, h is expressed as

$$h = ux^m, \tag{2.5}$$

where m is a positive integer and where u is a unit at P. By direct calculation, we have the following lemma.

Lemma 2.2. *If Δ is additive, then $m \not\equiv 1 \bmod p$. Moreover,* $\operatorname{ord}_P h_{ij} \geq i(m-1)+j$.

3. Numerical invariants of wild fibers

Let C (resp. E) be a non-singular complete curve (resp. a supersingular elliptic curve) over k. We set $X = E \times C$. Let Δ (resp. δ) be a non-zero additive vector field on C (resp. a non-zero regular vector field on E). Since E is supersingular, δ is additive. Let $\tilde{Q}_i$ $(i = 1, 2, \ldots, \lambda)$ be the zero points of Δ. We denote by m_i the order of zero of Δ at $\tilde{Q}_i$. We naturally extend δ and Δ to vector fields on X, which we denote by the same letters. We set $D = \delta + \Delta$. We note that, by Ganong and Russell [2, Lemma 3.7.1], any divisorial p-closed rational vector field on X can be normalized in this form in the case of $C = \mathbb{P}^1$. (In this normal form, either δ or Δ may be zero.) We set

$$S = X^D \quad \text{and} \quad B = C^{\Delta}. \tag{3.1}$$

Then, we have a diagram

$$\begin{array}{ccc} S & \xleftarrow{\ \pi\ } & X = E \times C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ B & \xleftarrow{\ F\ } & C, \end{array} \tag{3.2}$$

where pr is the second projection, where f is the morphism induced by pr, and where π and F are natural morphisms. The morphism F is nothing but the Frobenius morphism. In this section, we examine the elliptic surface $f: S = X^D \to B = C^{\Delta}$. We set $Q_i = F(\tilde{Q}_i)$, $(i = 1, 2, \ldots, \lambda)$. By Rudakov and Shafarevich [6, Proposition 1], the multiple fibers of f exist only over Q_i $(i = 1, \ldots, \lambda)$. By our construction, $f^{-1}(Q_i)$ is a multiple fiber of an elliptic curve E_i. By Rudakov and Shafarevich [6, Proposition 1], $\pi^{-1}(E_i)$ is reduced and $\pi^{-1}(E_i) = \mathrm{pr}^{-1}(\tilde{Q}_i)$. Since E is supersingular, E_i is also supersingular. We denote by d_i the multiplicity of $f^{-1}(Q_i)$. By $\pi^{-1}(f^{-1}(Q_i)) = \mathrm{pr}^{-1}(F^{-1}(Q_i))$ we have $d_i(\mathrm{pr}^{-1}(\tilde{Q}_i)) = p(\mathrm{pr}^{-1}(\tilde{Q}_i))$. Therefore, we have $d_i = p$ and $f^{-1}(Q_i) = pE_i$. Since the Picard variety $\mathrm{Pic}^0(E_i)$ has no points of order p, we have $\operatorname{ord} \mathcal{O}_S(E_i)|_{E_i} = 1$; hence, $f^{-1}(Q_i)$ is a wild fiber.

Lemma 3.1. $\chi(S, \mathcal{O}_S) = 0$.

Proof. Since π is radical, we have $c_2(S) = c_2(X)$. Since $c_2(X) = 0$, we have $c_2(S) = 0$. By Noether's formula, we have $\chi(S, \mathcal{O}_S) = (K_S^2 + c_2(S))/12 = 0$. $\square$

Take any point $\tilde{Q}$ among $\tilde{Q}_i$'s. We set $Q = F(\tilde{Q})$. Let x be a local parameter at $\tilde{Q}$. We denote by m the order of zero of Δ at $\tilde{Q}$. Then, Δ is expressed as

$$\Delta = ux^m \frac{\partial}{\partial x}, \tag{3.3}$$

where u is a unit at $\tilde{Q}$. As we see above, $f^{-1}(Q) = pF$ is a wild fiber. By Lemma 2.2, we have $m \geq 2$. We set $\pi^{-1}(F) = \tilde{E}$. Then, we have $\tilde{E} = \mathrm{pr}^{-1}(\tilde{Q})$. Since we have the natural isomorphism

$$H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}}) \simeq k[x]/(x^\ell), \tag{3.4}$$

we see that $\{1, x, \ldots, x^{\ell-1}\}$ is a basis of $H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}})$. Let z be a point on F in S, and $\tilde{z}$ a point on $\tilde{E}$ in X such that $\pi(\tilde{z}) = z$. Let Spec(R) be an affine open neighbourhood of $\tilde{z}$, and $\tilde{\boldsymbol{m}}$ the maximal ideal of R which corresponds to $\tilde{z}$. We set $A = R_{\tilde{\boldsymbol{m}}} \simeq (\mathcal{O}_X)_{\tilde{z}}$. We set $V = \mathrm{Spec}\, R^D$ and $\boldsymbol{m} = R^D \cap \tilde{\boldsymbol{m}}$. Then V is an affine open neighborhood of z, and $\boldsymbol{m}$ is the maximal ideal which corresponds to z. We easily see $A^D = (R^D)_{\boldsymbol{m}} = (\mathcal{O}_S)_z$. Let y be a local equation of F at z. Since $\pi^{-1}(F) = \tilde{E}$, we have $y = vx$ in A with a unit v on $\tilde{E}$. Then, we have

$$(\mathcal{O}_{\ell\tilde{E}})_{\tilde{z}} \simeq A/(x^\ell) \quad \text{and} \quad (\mathcal{O}_{\ell F})_z \simeq A^D/(y^\ell),$$

and we have the natural inclusion

$$A^D/(y^\ell) \subset A/(x^\ell). \tag{3.5}$$

By (3.3), we have

$$D(x^i) = iux^{m+i-1}.$$

Since $m \geq 2$ by the assumption on D, D induces a rational vector field on the subscheme $\ell\tilde{E}$, especially on Spec $A/(x^\ell)$, and we have

$$A^D/(y^\ell) \subset (A/(x^\ell))^D. \tag{3.6}$$

Therefore, we have

$$H^0(\ell F, \mathcal{O}_{\ell F}) \subsetneq (H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}}))^D. \tag{3.7}$$

We express the point $\tilde{z}$ of $X = E \times C$ as $\tilde{z} = (\tilde{z}_1, \tilde{Q})$ where $\tilde{z}_1 \in E$ and $\tilde{Q} \in C$. By Lemma 2.1, there exists a rational function f on E such that

$$\delta(f) = 1 \tag{3.8}$$

and such that f is regular at $\tilde{z}_1$. We naturally regard f as a rational function on X. We set for $1 \leq \alpha \leq p-1$

$$F_{i\alpha} = \left\{(-1)^\alpha \frac{1}{\alpha!} \sum_{\beta=1}^{\alpha} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma)\right) h_{\alpha\beta} x^{i-\beta}\right\} f^\alpha, \tag{3.9}$$

and

$$g_i = \sum_{\alpha=1}^{p-1} F_{i\alpha} + x^i. \tag{3.10}$$

Then, we have the following two lemmas.

Lemma 3.2. *Assume* $1 + \ell - m \le i \le \ell - 1$ *and* $i \ge 1$. *Then,*

$$F_{i\alpha} \in A \quad \text{and} \quad \operatorname{ord}_{\tilde{E}} F_{i\alpha} \ge (\alpha - 1)(m - 1) + \ell.$$

Proof. The former part is trivial. As for the latter part, by assumption we have

$$\operatorname{ord}_{\tilde{E}} F_{i\alpha} \ge \alpha(m-1) + \beta + i - \beta \ge (\alpha - 1)(m - 1) + \ell. \qquad \square$$

Lemma 3.3. *Assume* $1 + \ell - m \le i \le \ell - 1$ *and* $i \ge 1$. *Then,*

$$g_i = x^i \text{ in } A/(x^\ell) \quad \text{and} \quad g_i \in A^D.$$

Proof. The former part follows from Lemma 3.2. It is clear that $g_i \in A$. Using the notation in (2.1), by (3.8) we have

$$\begin{aligned}
\delta(g_i) &= \sum_{\alpha=1}^{p-1} \left\{ (-1)^\alpha \frac{1}{(\alpha-1)!} \sum_{\beta=1}^{\alpha} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) h_{\alpha\beta} x^{i-\beta} \right\} f^{\alpha-1} \\
&= \sum_{\alpha=0}^{p-2} \left\{ (-1)^{\alpha+1} \frac{1}{\alpha!} \sum_{\beta=1}^{\alpha+1} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) h_{\alpha+1\beta} x^{i-\beta} \right\} f^\alpha,
\end{aligned}$$

$$\begin{aligned}
h\frac{\partial g_i}{\partial x} &= h \left\{ \sum_{\alpha=1}^{p-1} \left((-1)^\alpha \frac{1}{\alpha!} \sum_{\beta=1}^{\alpha} \left(\prod_{\gamma=0}^{\beta} (i-\gamma) \right) h_{\alpha\beta} x^{i-\beta-1} \right.\right. \\
&\qquad \left.\left. + (-1)^\alpha \frac{1}{\alpha!} \sum_{\beta=1}^{\alpha} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) h'_{\alpha\beta} x^{i-\beta} \right) f^\alpha \right\} + ihx^{i-1} \\
&= \sum_{\alpha=1}^{p-1} \left\{ (-1)^\alpha \frac{1}{\alpha!} \left(\sum_{\beta=2}^{\alpha+1} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) hh_{\alpha,\beta-1} x^{i-\beta} \right.\right. \\
&\qquad \left.\left. + \sum_{\beta=1}^{\alpha} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) hh'_{\alpha\beta} x^{i-\beta} \right) f^\alpha \right\} + ihx^{i-1} \\
&= \sum_{\alpha=1}^{p-1} \left\{ (-1)^\alpha \frac{1}{\alpha!} \left(ih'_{\alpha 1} hx^{i-1} + \sum_{\beta=2}^{\alpha} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) h(h_{\alpha,\beta-1} + h'_{\alpha\beta}) x^{i-\beta} \right.\right. \\
&\qquad \left.\left. + \left(\prod_{\gamma=0}^{\alpha} (i-\gamma) \right) hh_{\alpha\alpha} x^{i-\alpha-1} \right) f^\alpha \right\} + ihx^{i-1} \\
&= \sum_{\alpha=1}^{p-1} \left\{ (-1)^\alpha \frac{1}{\alpha!} \left(ih_{\alpha+1,1} x^{i-1} + \sum_{\beta=2}^{\alpha} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) h_{\alpha+1,\beta} x^{i-\beta} \right.\right. \\
&\qquad \left.\left. + \left(\prod_{\gamma=0}^{\alpha} (i-\gamma) \right) h_{\alpha+1,\alpha+1} x^{i-\alpha-1} \right) f^\alpha \right\} + ihx^{i-1} \\
&= \sum_{\alpha=1}^{p-1} \left\{ (-1)^\alpha \frac{1}{\alpha!} \left(\sum_{\beta=1}^{\alpha+1} \left(\prod_{\gamma=0}^{\beta-1} (i-\gamma) \right) h_{\alpha+1,\beta} x^{i-\beta} \right) f^\alpha \right\} + ihx^{i-1}.
\end{aligned}$$

By $h_{11} = h$ and $h_{p\beta} = 0$ $(1 \le \beta \le p-1)$, we have

$$D(g_i) = (-1)ih_{11}x^{i-1} + (-1)^{p-1}\frac{1}{(p-1)!}\sum_{\beta=1}^{p}\left(\prod_{\gamma=0}^{\beta-1}(i-\gamma)\right)h_{p\beta}x^{i-\beta}f^{p-1} + ihx^{i-1}$$
$$= (-1)^{p-1}\frac{1}{(p-1)!}\left(\prod_{\gamma=0}^{p-1}(i-\gamma)\right)h_{pp}\,x^{i-p}f^{p-1}.$$

Since $\prod_{\gamma=0}^{p-1}(i-\gamma) \equiv 0 \bmod p$, we have

$$D(g_i) = 0;$$

hence, $g_i \in A^D$. □

We denote by V_1 the subspace of $H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}})$ spanned by x^{pi}'s $(pi \le \ell - 1, i \ge 0)$, and by V_2 the subspace of $H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}})$ spanned by x^i's $(1 + \ell - m \le i \le \ell - 1, i \ge 0, i \not\equiv 0 \bmod p)$. Since x^{pi}'s $(i \ge 0)$ are elements of A^D for each local ring A at any point of $\tilde{E}$, we see $V_1 \subset H^0(\ell F, \mathcal{O}_{\ell F})$. Since $D(x^i) = 0$ $(1 + \ell - m \le i \le \ell - 1)$ in $A/(x^\ell)$ by $m \ge 2$, we have $x^i \in (A/(x^\ell))^D$. By (3.6) and Lemma 3.3, we have $x^i \in A^D/(y^\ell)$. Therefore, we have $V_2 \subset H^0(\ell F, \mathcal{O}_{\ell F})$. Hence, we have

$$V_1 \oplus V_2 \subset H^0(\ell F, \mathcal{O}_{\ell F}) \subsetneq (H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}}))^D. \tag{3.11}$$

We denote by V_3 the subspace of $H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}})$ spanned by x^i's $(1 \le i \le l - m, i \not\equiv 0 \bmod p)$. Then, we have

$$H^0(\ell F, \mathcal{O}_{\ell F}) \oplus V_3 = H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}}) \quad\text{and}\quad V_3 \cap H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}})^D = 0. \tag{3.12}$$

By (3.11) and (3.12), we have the following lemma.

Lemma 3.4. $H^0(\ell F, \mathcal{O}_{\ell F}) \simeq V_1 \oplus V_2 = (H^0(\ell\tilde{E}, \mathcal{O}_{\ell\tilde{E}}))^D$.

For a real number α, we denote by $[\alpha]$ the integral part of α.

Theorem 3.5. *Under the notation above, the multiple fibers $f^{-1}(Q_i) = pE_i$ $(i = 1, 2, \dots, \lambda)$ are wild fibers, and we have:*

(1) $$\dim_k H^0(\ell F, \mathcal{O}_{\ell F}) = \begin{cases} \ell & \text{if } 1 \le \ell \le m, \\ m + \left[\dfrac{\ell - m}{p}\right] & \text{if } m < \ell. \end{cases}$$

(2) *The smallest positive integer n_1 such that* $\operatorname{ord}\mathcal{O}_S(F)|_{n_1F} = p$ *is equal to m.*

(3) $$K_S \sim f^*\left(K_C + \sum_{i=1}^{\lambda}\left[\frac{m_i(p-1)}{p}\right]Q_i\right) + \sum_{i=1}^{\lambda} a_iE_i,$$

where

$$a_i = \begin{cases} p - \left(m_i - \left[\frac{m_i}{p}\right]p\right) & \text{if } m_i \not\equiv 0 \bmod p, \\ 0 & \text{if } m_i \equiv 0 \bmod p. \end{cases}$$

Proof. As we have already shown, $f^{-1}(Q) = pF$ is a wild fiber.

(1) By definition, we have

$$\dim V_1 + \dim V_2 = \begin{cases} \ell & \text{if } 1 \le \ell \le m, \\ (\ell - 1) - (\ell - m) + 1 - \left[\frac{\ell - m}{p}\right] & \text{if } m < \ell. \end{cases}$$

Therefore, by Lemma 3.4, we complete the proof of (1).

(2) In the canonical divisor formula as in (1.1), we denote by a the a_i corresponding to the wild fiber pF. Then, we have

$$\omega_n = \mathcal{O}_S((a+n)F)|_{nF}$$

and

$$v = \operatorname{ord} \mathcal{O}_S(F)|_F = 1, \quad p = vp^1.$$

Therefore, in expression (1.4), we have $\gamma = 1$. Using the notation in Section 1, by the definition of n_1, we have

$$\operatorname{ord} \mathcal{O}_S(F)|_{nF} = 1 \quad \text{for } 1 \le n < n_1;$$

therefore, ω_n is trivial for $1 \le n < n_1$. Therefore, by Lemma 1.1, we have

$$\dim H^0(nF, \mathcal{O}_{nF}) = \dim H^0((n-1)F, \mathcal{O}_{(n-1)F}) + 1 \quad \text{for } 1 < n < n_1. \tag{3.13}$$

In case $n = n_1$, we have $\operatorname{ord} \mathcal{O}_S(F)|_{n_1F} = p$ by the definition of n_1. By Lemma 1.2, ω_{n_1} is trivial; therefore, we have by Lemma 1.1

$$\dim H^0(n_1F, \mathcal{O}_{n_1F}) = \dim H^0((n_1-1)F, \mathcal{O}_{(n_1-1)F}) + 1 \tag{3.14}$$

and $p \mid a + n_1$. Therefore, $p \nmid a + (n_1 + 1)$ and ω_{n_1+1} is not trivial. By Lemma 1.1, we have

$$\dim H^0((n_1+1)F, \mathcal{O}_{(n_1+1)F}) = \dim H^0(n_1F, \mathcal{O}_{n_1F}). \tag{3.15}$$

On the other hand, by the result in (1), we know

$$\dim H^0(\ell F, \mathcal{O}_{\ell F}) = \ell = \dim H^0((\ell-1)F, \mathcal{O}_{(\ell-1)F}) + 1 \quad \text{for } 1 < \ell \le m,$$

$$\dim H^0((m+1)F, \mathcal{O}_{(m+1)F}) = m + \left[\frac{(m+1) - m}{p}\right] = m = \dim H^0(mF, \mathcal{O}_{mF}). \tag{3.16}$$

Comparing (3.13), (3.14) and (3.15) with (3.16), we conclude $n_1 = m$.

(3) By Lemma 1.3, there exists a positive integer h such that $a = ph - n_1 = ph - m$. Since $0 \le a \le p-1$, we have

$$a = \begin{cases} p - \left(m - \left[\dfrac{m}{p}\right]p\right) & \text{if } m \not\equiv 0 \bmod p, \\ 0 & \text{if } m \equiv 0 \bmod p. \end{cases}$$

Now, we consider the canonical divisor formula of S as in (1.1). In $R^1 f_* \mathcal{O}_S \simeq \mathcal{L} \oplus \mathcal{T}$, we express $\mathcal{T}$ as

$$\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\lambda, \quad \operatorname{Supp} \mathcal{T}_i = Q_i \quad (i = 1, 2, \ldots, \lambda).$$

We set $t_i = \text{length}\, \mathcal{T}_i$. Then, we have length $\mathcal{T} = t = \sum_{i=1}^{\lambda} t_i$, and by Corollary 1.5 and (2), we have

$$t_i = \left[\frac{m_i(p-1)}{p}\right].$$

Therefore, we get the formula in (3). □

Corollary 3.6. $a_i \ne p-1$ $(i = 1, 2, \ldots, \lambda)$.

Proof. This follows from Theorem 3.5(3) and Lemma 2.2. □

Theorem 3.7. *Under the same notation as in Theorem* 3.5, *assume, moreover,* $C \simeq \mathbb{P}^1$. *Then,* $B \simeq \mathbb{P}^1$ *and the Frobenius mapping* F *on* $H^1(S, \mathcal{O}_S)$ *is the zero mapping.*

Proof. The fact $B \simeq \mathbb{P}^1$ is clear. In the diagram (3.2), take a general point P of B. Then, $f^{-1}(P) = G$ is an elliptic curve, and $\pi^{-1}(G) = p\tilde{E}$ with a general fiber $\tilde{E}$ of pr. Therefore, $\pi|_{\tilde{E}}: \tilde{E} \to G$ is an isomorphism. We set $h = \pi|_{\tilde{E}}$. We have a diagram

$$\begin{array}{ccc} G & \xleftarrow{h} & \tilde{E} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ S & \xleftarrow{\pi} & E \times \mathbb{P}^1, \end{array} \tag{3.17}$$

where g and $\tilde{g}$ are natural inclusions. By an exact sequence

$$0 \to \mathcal{O}_S(-G) \to \mathcal{O}_S \to \mathcal{O}_G \to 0,$$

we have a long exact sequence

$$\longrightarrow H^1(S, \mathcal{O}_S) \xrightarrow{g^*} H^1(G, \mathcal{O}_G) \longrightarrow H^2(S, \mathcal{O}_S(-G))$$
$$\longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow 0.$$

By the Serre duality $H^2(S, \mathcal{O}_S(-G)) \simeq H^{\circ}(S, \mathcal{O}_S(K_S + G))$,we have

$$\dim H^2(S, \mathcal{O}_S(-G)) = \dim H^2(S, \mathcal{O}_S) + 1.$$

Since $H^1(G, \mathcal{O}_G) \simeq k$, we see that g^* is a zero mapping. By the diagram (3.17), we have the diagram

$$\begin{array}{ccc} H^1(G, \mathcal{O}_G) & \xrightarrow{h^*} & H^1(\tilde{E}, \mathcal{O}_{\tilde{E}}) \\ \uparrow g^* & & \uparrow \tilde{g}^* \\ H^1(S, \mathcal{O}_S) & \xrightarrow{\pi^*} & H^1(E \times \mathbb{P}^1, \mathcal{O}_{E \times \mathbb{P}^1}). \end{array} \tag{3.18}$$

Since g^* is a zero mapping and h^*, $\tilde{g}^*$ are isomorphisms, we conclude that π^* is a zero mapping.

Now, we take an element α of $H^1(S, \mathcal{O}_S)$. Take an affine open covering $\{U_i\}_{i \in I}$ of S, and express α as a Čech cocycle $\{\alpha_{ij}\}_{i,j \in I}$ with respect to this covering. Set $V_i = \pi^{-1}(U_i)$. Since π is a finite morphism, $\{V_i\}_{i \in I}$ is an affine open covering of $E \times \mathbb{P}^1$. Since $\pi^*\alpha = 0$, there exists a regular function α_i on V_i such that

$$\pi^*(\alpha_{ij}) = \alpha_j - \alpha_i.$$

Therefore, we have $\pi^*(\alpha_{ij}^p) = \alpha_j^p - \alpha_i^p$. Since $\alpha_i^p \in k(E \times \mathbb{P}^1)^p$ and $k(E \times \mathbb{P}^1)^p \subset k(S)$, there exists a rational function β_i on U_i such that $\pi^*\beta_i = \alpha_i^p$. Since α_i is regular on V_i, β_i is also regular on U_i. Hence, we have

$$\alpha_{ij}^p = \beta_j - \beta_i$$

with a regular function β_i on U_i $(i \in I)$. This means that $\boldsymbol{F}(\alpha) = 0$ in $H^1(S, \mathcal{O}_S)$. □

4. A characterization of certain elliptic surfaces

Let $f: S \to \mathbb{P}^1$ be an elliptic surface which has the multiple fibers pE_i $(i = 1, 2, \dots, \lambda;\ \lambda \geq 1)$ with multiplicity p. We set $f(pE_i) = Q_i$ $(i = 1, \dots, \lambda)$. We assume

$$\chi(S, \mathcal{O}_S) = 0. \tag{4.1}$$

Then, as is easily proved, f has no degenerate fibres except over $f(pE_i) = Q_i$ $(i = 1, \dots, \lambda)$. We fix a general point P of $\mathbb{P}^1$. A canonical divisor of S is given by

$$K_S \sim f^*((-2 + t)P) + \sum_{i=1}^{\lambda} a_i E_i, \tag{4.2}$$

where t is the length of the torsion part of $R^1 f_* \mathcal{O}_S$, and $0 \leq a_i \leq p - 1$ $(i = 1, 2, \dots, \lambda)$. We assume two more conditions:

$$a_i \neq p - 1 \quad (i = 1, 2, \dots, \lambda). \tag{4.3}$$

The Frobenius mapping $\boldsymbol{F}$ on $H^1(S, \mathcal{O}_S)$ is the zero mapping. (4.4)

Lemma 4.1. *Under the assumptions* (4.1), (4.3) *and* (4.4), *all multiple fibers pE_i are wild, and any fiber of f is either a supersingular elliptic curve or a multiple fiber of a supersingular elliptic curve.*

Proof. By $\lambda \geq 1$, we have some multiple fibers. Since $a_i \neq p-1$, pE_i is a wild fiber. By the exact sequence

$$0 \to \mathcal{O}_S(-E_i) \to \mathcal{O}_S \to \mathcal{O}_{E_i} \to 0,$$

we have a long exact sequence

$$\longrightarrow H^1(S, \mathcal{O}_S) \xrightarrow{r} H^1(E_i, \mathcal{O}_{E_i}) \longrightarrow H^2(S, \mathcal{O}_S(-E_i))$$
$$\longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow 0.$$

By the Serre duality and $a_i \neq p-1$, we have

$$H^2(S, \mathcal{O}_S(-E_i)) \simeq H^0(S, \mathcal{O}_S(K_S + E_i)) \simeq H^0(S, \mathcal{O}_S(K_S)) \simeq H^2(S, \mathcal{O}_S).$$

Since $H^1(E_i, \mathcal{O}_{E_i}) \simeq k$, we see that r is surjective. By assumption, the action of the Frobenius mapping F on $H^1(S, \mathcal{O}_S)$ is trivial, and so is the action on $H^1(E_i, \mathcal{O}_{E_i})$. Therefore, E_i is a supersingular elliptic curve.

Since $\chi(S, \mathcal{O}_S) = 0$, we have $c_2(S) = 2 - 4q(S) + b_2(S) = 0$ by Noether's formula. Therefore, we have $q(S) \geq 1$. Since the base curve is $\mathbb{P}^1$, we have $q(S) \leq 1$ (cf. [4, Lemma 3.4]). Therefore, $q(S) = 1$ and the Albanese mapping $\psi: S \to \mathrm{Alb}(S)$ is surjective (cf. [4, Lemma 3.4]). Therefore, for any fiber $f^{-1}(P)$ ($P \in \mathbb{P}^1$), the restriction of ψ on $f^{-1}(P)$ is surjective. This means that E_i is isogenous to $\mathrm{Alb}(S)$ and that any regular fiber is also isogenous to $\mathrm{Alb}(S)$. Hence, any regular fiber is also supersingular. □

Theorem 4.2. *Let $f: S \to \mathbb{P}^1$ be an elliptic surface with multiple fibers pE_i ($i = 1, 2, \ldots, \lambda$; $\lambda \geq 1$) with multiplicity p. Assume that the elliptic surface satisfies the conditions* (4.1), (4.3) *and* (4.4). *Then, the elliptic surface is constructed as in* (3.1) *with $C = \mathbb{P}^1$. Namely, there exist a supersingular elliptic curve, a non-zero regular vector field δ on E and a non-zero rational vector field Δ on $\mathbb{P}^1$ such that $S = (E \times \mathbb{P}^1)^D$ with $D = \delta + \Delta$ and such that $f: S \to (\mathbb{P}^1)^\Delta = \mathbb{P}^1$ is the natural projection.*

Proof. We take a point P_i among P_i's ($i = 1, 2, \ldots, \lambda$). We denote it by P_0, and we set $pE_0 = f^{-1}(P_0)$. Let x be a local coordinate of an affine line $\mathbb{A}^1$ in $\mathbb{P}^1$. We may assume that $x = 0$ defines the point P_0, and that the fiber E_∞ over the point P_∞ at infinity is a regular fiber. Multiplying x, we have an isomorphism

$$\times x\colon \mathcal{O}_S(-E_\infty) \simeq \mathcal{O}_S(-pE_0).$$

Therefore, we have an isomorphism

$$\times x\colon H^1(S, \mathcal{O}_S(-E_\infty)) \simeq H^1(S, \mathcal{O}_S(-pE_0)). \tag{4.5}$$

By the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-E_\infty) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{E_\infty} \longrightarrow 0,$$

we have a long exact sequence

$$0 \longrightarrow H^1(S, \mathcal{O}_S(-E_\infty)) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow H^1(E_\infty, \mathcal{O}_{E_\infty})$$
$$\longrightarrow H^2(S, \mathcal{O}_S(-E_\infty)) \longrightarrow H^2(S, \mathcal{O}_S) \longrightarrow 0.$$

Since $\dim H^2(S, \mathcal{O}_S(-E_\infty)) = \dim H^2(S, \mathcal{O}_S) + 1$ and $\dim H^1(E_\infty, \mathcal{O}_{E_\infty}) = 1$, we have an isomorphism

$$H^1(S, \mathcal{O}_S(-E_\infty)) \simeq H^1(S, \mathcal{O}_S). \tag{4.6}$$

By (4.5) and (4.6), we have an isomorphism

$$\varphi : H^1(S, \mathcal{O}_S) \simeq H^1(S, \mathcal{O}_S(-E_\infty)) \simeq H^1(S, \mathcal{O}_S(-pE_0)). \tag{4.7}$$

Considering the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_S(-E_0) & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_{E_0} & \longrightarrow & 0 \text{ (exact)} \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_S(-pE_0) & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_{pE_0} & \longrightarrow & 0 \text{ (exact)}, \end{array}$$

we have a diagram

$$\begin{array}{ccccccc} H^1(S, \mathcal{O}_S(-E_0)) & \longrightarrow & H^1(S, \mathcal{O}_S) & \xrightarrow{\rho} & H^1(S, \mathcal{O}_{E_0}) & \longrightarrow & 0 \\ \uparrow & & \| & & \uparrow & & \\ H^1(S, \mathcal{O}_S(-pE_0)) & \xrightarrow{\phi} & H^1(S, \mathcal{O}_S) & \xrightarrow{\rho_1} & H^1(S, \mathcal{O}_{pE_0}), & & \end{array} \tag{4.8}$$

where the first and the second rows are exact by the assumption $a_i \neq p-1$. Using (4.7) and (4.8), we have a homomorphism

$$\phi \circ \varphi : H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \mathcal{O}_S(-E_\infty)) \xrightarrow{\times x} H^1(S, \mathcal{O}_S(-pE_0))$$
$$\xrightarrow{\phi} \quad H^1(S, \mathcal{O}_S). \tag{4.9}$$

We know $H^1(S, \mathcal{O}_{E_0}) \simeq k$. Take an element α of $H^1(S, \mathcal{O}_S)$ such that $\rho(\alpha) \neq 0$. We consider elements of $H^1(S, \mathcal{O}_S)$ given by

$$\alpha, \phi \circ \varphi(\alpha), (\phi \circ \varphi)^2(\alpha), \ldots, (\phi \circ \varphi)^n(\alpha). \tag{4.10}$$

Since $H^1(S, \mathcal{O}_S)$ is finite-dimensional, the elements of (4.10) are linearly dependent if n is large enough. We take the smallest integer n such that the elements of (4.10) are linearly dependent. Then, there exists $(b_0, b_1, \ldots, b_n) \in k^{n+1}$ such that

$$b_0\alpha + b_1\phi \circ \varphi(\alpha) + \cdots + b_n(\phi \circ \varphi)^n(\alpha) = 0 \quad \text{and} \quad (b_0, b_1, \ldots, b_n) \neq 0. \tag{4.11}$$

Suppose $b_0 \neq 0$. Then, we have

$$\alpha = -\phi\left\{\frac{b_1}{b_0}\varphi(\alpha) + \cdots + \frac{b_n}{b_0}\varphi \circ (\phi \circ \varphi)^{n-1}(\alpha)\right\}.$$

Therefore, we have $\rho(\alpha) = 0$. A contradiction. Therefore, we have $b_0 = 0$. We take the smallest integer ℓ such that $b_\ell \neq 0$. We set

$$\beta = b_\ell\alpha + b_{\ell+1}\phi \circ \varphi(\alpha) + \cdots + b_n(\phi \circ \varphi)^{n-\ell}(\alpha).$$

Since $b_\ell \neq 0$, we have

$$\rho(\beta) = \rho(b_\ell\alpha) \neq 0, \tag{4.12}$$

and by (4.11) we have

$$(\phi \circ \varphi)^\ell(\beta) = 0. \tag{4.13}$$

We take an affine open covering $\{U_i\}_{i\in I}$ of S, and we represent β by a Čech cocycle $\beta = \{\beta_{ij}\}$ $(\beta_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_S))$ with respect to $\{U_i\}_{i\in I}$. By (4.13), there exist $\beta_i \in \Gamma(U_i, \mathcal{O}_S)$ $(i \in I)$ such that

$$x^\ell \beta_{ij} = \beta_j - \beta_i \quad \text{on } U_i \cap U_j \ (i, j \in I). \tag{4.14}$$

On the other hand, by the assumption (4.4), there exist $h_i \in \Gamma(U_i, \mathcal{O}_S)$ $(i \in I)$ such that

$$\beta_{ij}^p = h_j - h_i \quad \text{on } U_i \cap U_j \ (i, j \in I). \tag{4.15}$$

By (4.14) and (4.15), we have

$$h_i - \left(\frac{\beta_i}{x^\ell}\right)^p = h_j - \left(\frac{\beta_j}{x^\ell}\right)^p \quad \text{on } U_i \cap U_j \ (i, j \in I). \tag{4.16}$$

We set

$$h = h_i - \left(\frac{\beta_i}{x^\ell}\right)^p \quad \text{on } U_i \ (i \in I). \tag{4.17}$$

Then, by (4.16) h is a rational function on S. The pole of h exists only on E_0. Since $f_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^1}$, we see that there exists a rational function $g(x)$ on $\mathbb{P}^1$ such that

$$f^*(g(x)) = h. \tag{4.18}$$

We set

$$\omega = dh_i \quad \text{on} \quad U_i \ (i \in I). \tag{4.19}$$

Since β is not zero in $H^1(S, \mathcal{O}_S)$, by (4.15) ω is a non-zero 1-form on S. Moreover, by (4.17) and (4.18) we have

$$\omega = f^*(dg(x)). \tag{4.20}$$

Now, we consider the purely inseparable covering X of S of degree p defined by

$$\begin{cases} z_i^p = h_i & \text{on } U_i \quad (i \in I), \\ z_j = z_i + \beta_{ij} & \text{on } U_i \cap U_j \quad (i, j \in I). \end{cases} \tag{4.21}$$

We denote by π the natural morphism $X \to S$. Since $\beta \neq 0$, this covering is not trivial. By (4.19) and (4.21), we have $\pi^*\omega = 0$. Therefore, by (4.20) we have $d(\pi^* f^*(g(x))) = 0$. Therefore, there exists a rational function $\tilde{g}$ on X such that

$$\tilde{g}^p = \pi^* f^*(g(x)).$$

This means that the base curve $\mathbb{P}^1$ is not algebraically closed in the function field $k(X)$. Considering the normalization of this base curve $\mathbb{P}^1$ in $k(X)$, we have the following diagram:

$$\begin{array}{ccccccc} S & \xleftarrow{\tilde{F}} & \tilde{S} & \xleftarrow{\mu} & X & \xleftarrow{\nu} & \tilde{X} \\ f\downarrow & & \downarrow f' & & & & \\ \mathbb{P}^1 & \xleftarrow{F} & \mathbb{P}^1, & & & & \end{array} \tag{4.22}$$

where $\tilde{S}$ is the fiber product of S and $\mathbb{P}^1$ over $\mathbb{P}^1$, where ν is the normalization of X, and $\pi = \tilde{\boldsymbol{F}} \circ \mu$. Since π is a purely inseparable morphism of degree p, we see that $\boldsymbol{F}$ is also a purely inseparable morphism of degree p, that is, the Frobenius morphism. We set

$$\tilde{f} = f' \circ \mu \circ \nu.$$

Since any fiber of f is either an elliptic curve or a multiple fiber of an elliptic curve, we see that $\tilde{X}$ is non-singular. By (4.12), the restriction of the covering π on E_0 is non-trivial. Therefore, $\nu^{-1} \circ \pi^{-1}(E_0)$ is a regular fiber of $\tilde{f}: \tilde{X} \to \mathbb{P}^1$. On the other hand, $\tilde{f}: \tilde{X} \to \mathbb{P}^1$ is constructed by using the base change by the Frobenius mapping $\boldsymbol{F}: \mathbb{P}^1 \to \mathbb{P}^1$ as in the diagram (4.22). Therefore, we conclude that $\tilde{f}: \tilde{X} \to \mathbb{P}^1$ has no multiple fibers. Therefore, this elliptic surface has no degenerate fibers. Hence, as is well known, $\tilde{X}$ is isomorphic to $E \times \mathbb{P}^1$ with an elliptic curve E, and $\tilde{f}$ is the second projection. By Lemma 4.1, E must be supersingular. Since $\pi \circ \nu$ is radical, by the standard theory of vector field in positive characteristic we complete our proof (cf. [2, Section 3]). □

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